The above result for the specific heat suggests that fluctuations of the order-parameter become strong for $d < 4$ and therefore the mean-field theory cannot be trusted. Let us argue based on the importance of terms neglected within the mean-field theory.

In the mean-field theory, we replaced

$$S_i S_j = [\langle S_i \rangle + \delta S_i][\langle S_j \rangle + \delta S_j \rangle$$

by

$$\langle S_i \rangle \langle S_j \rangle + \langle S_i \rangle \delta S_j + \langle S_j \rangle \delta S_j$$

That is, we dropped the term quadratic in fluctuations. The fractional error made is

$$\text{Error} = \frac{|\langle \delta S_i \, \delta S_j \rangle|}{|\langle S_i \rangle \langle S_j \rangle|} = \frac{|\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle|}{|\langle S_i \rangle| \, |\langle S_j \rangle|}$$

$$\simeq \frac{\int G_c(r) \, d^d r}{\int d^d r \, m^2}$$

where $G_c(r) = \langle S(r) S(0) \rangle - \langle S(r) \rangle \langle S(0) \rangle$

and the integration is

restricted to a region of the order of the correlation length.

The numerator is approximately,

$$\int \frac{d^d r}{r^{d-2+\eta}} f\left(\frac{r}{\xi}\right) \sim \xi^{2-\eta} \sim |t|^{-\nu(2-\eta)}$$
$$= |t|^{-\gamma}$$

The denominator is approximately

$$m^2 \xi^d \sim |t|^{2\beta - d\nu}$$

For the error to be small, one requires

$$|t|^{-\gamma} \ll |t|^{2\beta - d\nu} \quad \text{as } t \to 0$$

or $\quad 2\beta - d\nu + \gamma < 0$

$$\Rightarrow \quad d > \frac{2\beta + \gamma}{\nu}$$

This dimension, above which mean-field holds, is called 'upper critical dimension' d_L.

For the Ising theory, the mean-field exponents are $\beta = \frac{1}{2}$, $\gamma = 1$, $\nu = \frac{1}{2}$ $\Rightarrow$ $d_L = 4$, as anticipated.

The above analysis suggests that for $d<4$, one would never observe mean-field behavior. This is actually an incorrect conclusion, because the range of temperature over which the above criteria for mean-field to break down can in practice be quite small. To see this, lets keep all the dimensions in the above analysis, instead of doing just the scaling analysis.

The numerator is given by $\int d^d r \; G(r) \sim \chi$ where χ is the susceptibility $\sim |t|^{-1}$ if the Landau free energy $f \sim t m^2 + u m^4$. This doesn't change the above analysis for the numerator. The denominator is more interesting.

$$m^2 \xi^d \sim \left|\frac{t}{u}\right| \xi^d \sim \left|\frac{t}{u}\right| \xi(1)^d \, t^{-d\nu}$$

$$\sim \frac{\xi(1)^d}{u} |t|^{1-\frac{d}{2}}$$ where $\xi(1)$ is the 'bare correlation length' i.e. the length scale over which the d.o.f.'s are correlated away from the critical point when $|t| \sim O(1)$.

Thus, for mean field to hold, one requires,

$$\frac{1}{|t|} \ll \frac{\xi(1)^d \, |t|^{1-d/2}}{u}$$

or,
$$|t|^{2-d/2} \gg \frac{u}{\xi(1)^d}$$

Therefore, if the bare correlation length is large compared to lattice spacing, then the above condition may be satisfied even for $d < 4$ when $|t|$ is small. This is indeed the case in weakly-coupled BCS superconductors where $\frac{\xi}{a} \sim 10^3$. Thus, in such a system, one needs $|t| \sim 10^{-9\times 2} \sim 10^{-18}$ in $d = 3$. Therefore, for all practical considerations, one will not be able to observe the non-mean-field critical exponents. In contrast, in magnets, superfluids and high-T_c superconductors, $\frac{\xi}{a} \sim O(1)$ and one typically observes non-mean field behaviour as soon as $|t| \sim O(1)$.